

Relativistic analysis of the dielectric Einstein box: Abraham, Minkowski and total energy-momentum tensors

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Abstract

We analyse the “Einstein box” thought experiment and the definition of the momentum of light inside matter. We stress the importance of the total energy-momentum tensor of the closed system (electromagnetic field plus material medium) and derive in detail the relativistic expressions for the Abraham and Minkowski momenta, together with the corresponding balance equations for an isotropic and homogeneous medium. We identify some assumptions hidden in the Einstein box argument, which make it weaker than it is usually recognized. In particular, we show that the Abraham momentum is not uniquely selected as the momentum of light in this case.

Keywords: Electrodynamics, Abraham-Minkowski controversy, Energy-momentum tensor, Relativity, Einstein box experiment

1. Introduction

The problem of the adequate definition of the momentum of light inside material media has surprisingly been discussed for more than 100 years since the original papers of Minkowski [1] and Abraham [2], and even so there is still some confusion or at least disagreement among authors. Minkowski proposed a non-symmetric energy-momentum tensor, which has the advantages that it can be directly derived from Maxwell’s macroscopic equations and that it is related to the symmetries of the medium, when the latter is fixed (without dynamics). On the other hand, the Abraham tensor is symmetric, but cannot be derived from first principles [3]. The Minkowski and Abraham momentum densities of a light pulse in a medium at rest, are defined as

$$\pi_M := D \times B, \quad \pi_A := \frac{1}{c^2} E \times H. \quad (1)$$

In the simple case of plane waves propagating within an isotropic and homogeneous medium at rest, the rival expressions (1) reduce to:

$$\pi_M = n \frac{\mathcal{U}}{c} \hat{k}, \quad \pi_A = \frac{1}{n} \frac{\mathcal{U}}{c} \hat{k}, \quad (2)$$

where n is the refractive index of the medium, \mathcal{U} is the energy density and \hat{k} is the propagation unit vector. The difference between the very idealized predictions (2) motivated the debate of determining which of the two definitions for the momentum density of light inside media is the correct one. In the literature we can find many theoretical discussions and a few

experiments, which seem to favor one, both or neither of the two momenta for light discussed here. For a good and concise review of the Abraham-Minkowski controversy see the introduction of [4] and for a more detailed and historical review, not always free of confusion and contradictions, see [5]-[8].

The fundamentals of the controversy were understood in a formal manner more than 40 years ago, see [9]-[15]. Basically, it was recognized that only the total energy-momentum of the closed system consisting of electromagnetic field plus material medium has absolute physical meaning and that the Minkowski, Abraham and other expressions for the electromagnetic field simply correspond to different separations of the same total tensor. The expression for the total tensor will of course depend on the nature of the specific medium.

In the past 10 years, the discussion of the momentum of light inside media has become relevant again, specially due to the large number of more practical and “quantum oriented” works of Loudon, Barnett and collaborators, which analyze the radiation pressure based on the Lorentz force, [16]-[20]. Mansuripur does a similar analysis, but using a modified force definition [21, 22]. In this group of papers, it is usually stated that the so-called “Einstein box theories”, first proposed by Balazs [23] as a modified Einstein thought experiment [24], uniquely select the Abraham momentum as the momentum of the field. Their main arguments are that the Minkowski momentum would predict a motion of the slab in the opposite direction to the incident pulse and that the Abraham momentum is the only one which simultaneously conserves the velocity of the center of energy, the total energy and the total momentum of the system. Here we explicitly show that using the Minkowski momentum with its adequate balance equations, i.e. with the total energy-momentum tensor, one arrives at the same results as with the Abraham momentum. Most recently, Barnett and Loudon in [25, 26] reanalyzed the controversy and argued that both mo-

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menta are “correct”, because both could be measured, but in different situations. They identify the Abraham momentum as the “kinetic” momentum of light and use Balazs’s idea as their strongest argument to discard the Minkowski momentum in that situation. In other types of experiments, where the medium is at rest, it is claimed that the Minkowski momentum correctly describes the situation and therefore identify it as the “canonical” momentum of the field. In [27] it is also claimed that the controversy was solved, but in a different manner. One difference is that the author recognizes that the Einstein box argument does not uniquely determine the momentum of the field, but despite that he insists that this is the strongest available argument for the identification of light’s momentum inside media and therefore considers the definition of the Abraham momentum for the field as an additional postulate of his theory.

In this letter we make use of the general result given in [3] for the total energy-momentum of the system consisting of electromagnetic field and an isotropic dielectric medium and derive, in a completely relativistic and self-consistent manner, the expressions for the Abraham and Minkowski momenta together with the corresponding balance equations for the case of a light pulse propagating inside a dielectric slab. This approach will allow us to identify some assumptions hidden in the Einstein box argument, which make it weaker than it is usually recognized.

2. Relativistic model and the total energy-momentum tensor

Suppose there is a dielectric slab of mass M with homogeneous and isotropic electromagnetic properties, floating in space. In its rest frame, its index of refraction is n , its length is L and it occupies a finite volume V . The slab is initially at rest, but a light pulse of total energy \mathcal{E}_0 and finite volume $V_p \ll V$ strikes the slab from vacuum at normal incidence putting it in motion with a *final* constant velocity v . The slab is equipped with anti-reflection coatings so that the pulse can enter the slab at normal incidence without reflection and energy losses.

A fully relativistic model for the total energy-momentum tensor T_{μ}^{ν} for a general linear, non-dissipative, non-dispersive and isotropic dielectric fluid with proper energy density ρ , pressure p , 4-velocity field u^{μ} , relative permittivity ε , relative permeability μ and particle number density ν , interacting with the electromagnetic field $F_{\mu\nu}$ was first derived by Penfield and Haus in 1966 [9] and is explicitly given in more modern form in the review [3]. Following the same conventions as in [3], and neglecting gravitational as well as possible electro- and magnetostriction effects, and assuming negligible pressure $p \approx 0$, we have

$$T_{\mu}^{\nu} = \frac{\rho}{c^2} u_{\mu} u^{\nu} + \frac{1}{\mu\mu_0} \left(F_{\mu\sigma} F^{\sigma\nu} + \frac{1}{4} \delta_{\mu}^{\nu} F^{\sigma\lambda} F_{\sigma\lambda} \right) + \frac{(n^2 - 1)}{\mu\mu_0 c^2} \left(F_{\mu\sigma} F^{\lambda\nu} u^{\sigma} u_{\lambda} + \frac{1}{2} \delta_{\mu}^{\nu} F_{\sigma\rho} F^{\sigma\lambda} u^{\rho} u_{\lambda} - \frac{1}{c^2} F^{\rho\sigma} F_{\rho\lambda} u_{\sigma} u^{\lambda} u_{\mu} u^{\nu} \right). \quad (3)$$

The total tensor is symmetric and satisfies the following energy-momentum balance equation,

$$\partial_{\nu} T_{\mu}^{\nu} - F_{\mu\nu} J_{\text{ext}}^{\nu} = 0, \quad (4)$$

where the 4-vector J_{ext}^{ν} describes the external charge and current densities which do not belong to the dielectric fluid. If $J_{\text{ext}}^{\nu} = 0$ energy-momentum tensor of the complete system is conserved and we have a closed system.

If we choose a volume V' big enough so that it encloses the pulse and the slab until the pulse leaves the slab from the other side, then we can integrate the conservation equation and obtain that the total 4-momentum $\mathcal{P}_{\mu} := (E, -\mathbf{p})$ of the whole system, defined as

$$\mathcal{P}_{\mu} := \int_{V'} T_{\mu}^0 dV, \quad (5)$$

is a conserved, i.e. time independent, quantity. We will use this conservation of energy and momentum of the closed system to study the motion of the slab, when the pulse is propagating inside it (once the slab achieved a final constant velocity after a short deformation transient).

Because of the anti-reflection coatings, the light pulse can pass completely in the same incident direction (without reflection components), so that the problem can be treated just as a one-dimensional problem. Therefore the 4-velocity $u^{\mu} := (\gamma, \gamma v)$ of the dielectric slab can be chosen to be

$$u^{\mu} = (\gamma, \gamma v, 0, 0), \quad (6)$$

where $\gamma := (1 - \beta^2)^{-1/2}$ and $\beta := v/c$ as usual. Finally, we assume that the energy density distribution in the comoving frame is homogeneous:

$$\rho = \frac{Mc^2}{V} = \text{const.} \quad (7)$$

3. Energy-momentum tensors of the electromagnetic field

The total energy-momentum tensor (3) can be split in different ways. For example, we can assign for the slab the energy-momentum tensor of a fluid without pressure (dust):

$$\overset{\text{m}}{\Omega}_{\mu}^{\nu} := \frac{\rho}{c^2} u_{\mu} u^{\nu}, \quad (8)$$

and then light will be described by the Abraham energy-momentum tensor [3]

$$\Omega_{\mu}^{\nu} := T_{\mu}^{\nu} - \overset{\text{m}}{\Omega}_{\mu}^{\nu} \quad (9)$$

$$= \frac{1}{\mu\mu_0} \left(F_{\mu\sigma} F^{\sigma\nu} + \frac{1}{4} \delta_{\mu}^{\nu} F^{\sigma\lambda} F_{\sigma\lambda} \right) + \frac{(n^2 - 1)}{\mu\mu_0 c^2} \left(F_{\mu\sigma} F^{\lambda\nu} u^{\sigma} u_{\lambda} + \frac{1}{2} \delta_{\mu}^{\nu} F_{\sigma\rho} F^{\sigma\lambda} u^{\rho} u_{\lambda} - \frac{1}{c^2} F^{\rho\sigma} F_{\rho\lambda} u_{\sigma} u^{\lambda} u_{\mu} u^{\nu} \right). \quad (10)$$

With this interpretation the total conserved 4-momentum of the system in (5) turns out to be $\mathcal{P}_\mu = \overset{\text{m,A}}{\mathcal{P}}_\mu + \overset{\text{A}}{\mathcal{P}}_\mu$, where

$$\overset{\text{m,A}}{\mathcal{P}}_\mu := \int_{V'} \overset{\text{m}}{\Omega}_\mu^0 dV, \quad \overset{\text{A}}{\mathcal{P}}_\mu := \int_{V'} \overset{\text{A}}{\Omega}_\mu^0 dV. \quad (11)$$

On the other hand, we can consider that the electromagnetic energy and momentum content is described by the Minkowski energy-momentum tensor Θ_μ^ν , whose definition follows directly from Maxwell's equations [3],

$$\Theta_\mu^\nu := F_{\mu\sigma} H^{\sigma\nu} + \frac{1}{4} \delta_\mu^\nu F_{\sigma\lambda} H^{\sigma\lambda} \quad (12)$$

$$= \frac{1}{\mu\mu_0} \left(F_{\mu\sigma} F^{\sigma\nu} + \frac{1}{4} \delta_\mu^\nu F_{\sigma\lambda} F^{\sigma\lambda} \right) + \frac{(n^2 - 1)}{\mu\mu_0 c^2} \left(F_{\mu\lambda} F^{\lambda\rho} u_\rho u^\nu + F_{\mu\sigma} F^{\lambda\nu} u^\sigma u_\lambda + \frac{1}{2} \delta_\mu^\nu F_{\sigma\rho} F^{\sigma\lambda} u^\rho u_\lambda \right). \quad (13)$$

This tensor can be obtained by adding the term

$$Q_\mu^\nu := \frac{(n^2 - 1)}{\mu\mu_0 c^2} \left[F_{\mu\lambda} F^{\lambda\rho} u_\rho u^\nu + \frac{1}{c^2} F^{\rho\sigma} F_{\rho\lambda} u_\sigma u^\lambda u_\mu u^\nu \right] \quad (14)$$

to the Abraham tensor, so that

$$\Theta_\mu^\nu = \Omega_\mu^\nu + Q_\mu^\nu. \quad (15)$$

Consequently, the total energy-momentum tensor can be written as $T_\mu^\nu = \overset{\text{m}}{\Theta}_\mu^\nu + \Theta_\mu^\nu$, where

$$\overset{\text{m}}{\Theta}_\mu^\nu := \overset{\text{m}}{\Omega}_\mu^\nu - Q_\mu^\nu \quad (16)$$

$$= \frac{\rho}{c^2} u_\mu u^\nu - \frac{(n^2 - 1)}{\mu\mu_0 c^2} \left[F_{\mu\lambda} F^{\lambda\rho} u_\rho u^\nu + \frac{1}{c^2} F^{\rho\sigma} F_{\rho\lambda} u_\sigma u^\lambda u_\mu u^\nu \right], \quad (17)$$

is the Minkowski energy-momentum tensor for matter. Finally, with this interpretation, the total conserved 4-momentum \mathcal{P}_μ can be also expressed as $\mathcal{P}_\mu = \overset{\text{m,M}}{\mathcal{P}}_\mu + \overset{\text{M}}{\mathcal{P}}_\mu$, where

$$\overset{\text{m,M}}{\mathcal{P}}_\mu := \int_{V'} \overset{\text{m}}{\Omega}_\mu^0 dV, \quad \overset{\text{M}}{\mathcal{P}}_\mu := \int_{V'} \overset{\text{M}}{\Omega}_\mu^0 dV, \quad (18)$$

are the Minkowski 4-momenta for matter and electromagnetic field, respectively.

4. Explicit calculation with the Abraham tensor

4.1. Abraham energy and momentum for the slab

We derive first the Abraham tensor, which is more compact in this case. If we substitute (6) into (11a) and use the identification $\overset{\text{m,A}}{\mathcal{P}}_\mu = (\overset{\text{m,A}}{E}, -\overset{\text{m,A}}{\mathbf{p}})$, then the Abraham energy and momentum for the slab read

$$\overset{\text{m,A}}{E} = \int_{V'} \rho \frac{u_0 u^0}{c^2} dV, \quad (19)$$

$$\overset{\text{m,A}}{p}_a = - \int_{V'} \rho \frac{u_a u^0}{c^2} dV, \quad a = 1, 2, 3. \quad (20)$$

Therefore, by explicitly calculating the integrals, we obtain

$$\overset{\text{m,A}}{E} = \rho \gamma^2 V_\nu, \quad \overset{\text{m,A}}{p}_a = - \frac{1}{c^2} g_{ab} v^b \rho \gamma^2 V_\nu, \quad (21)$$

where V_ν is the volume of the slab in the reference frame where it moves with velocity $\mathbf{v} = v \hat{\mathbf{x}}$. Using the relation $V_\nu = V/\gamma$ and expression (7), we have

$$\overset{\text{m,A}}{E} = \gamma M c^2, \quad \overset{\text{m,A}}{\mathbf{p}} = \gamma M v \hat{\mathbf{x}}. \quad (22)$$

These results (22) are the usual expressions for the (relativistic) energy and momentum of a body of mass M moving with velocity $v \hat{\mathbf{x}}$, which is not surprising because of our choice (8) for the energy and momentum of the medium. The relation between momentum and energy is also the usual one for a relativistic massive particle:

$$\overset{\text{m,A}}{\mathbf{p}} = v \frac{\overset{\text{m,A}}{E}}{c^2} \hat{\mathbf{x}}. \quad (23)$$

4.2. Abraham energy and momentum for the light pulse

Using (6), (10), (11b) and the identifications of the components of $F_{\mu\nu}$ in [3], we can explicitly compute the energy and momentum associated to the Abraham tensor for the electromagnetic field in terms of \mathbf{E} , \mathbf{B} and \mathbf{v} :

$$\begin{aligned} \overset{\text{A}}{E} &= \frac{1}{2\mu\mu_0} \int_{V'} (\mathbf{E}^2/c^2 + \mathbf{B}^2) dV \\ &\quad - \frac{(n^2 - 1)}{2\mu\mu_0 c^2} \int_{V'} \left\{ \gamma^2 (2\gamma^2 - 1) [(\mathbf{E} \cdot \mathbf{v})^2/c^2 + (\mathbf{B} \cdot \mathbf{v})^2 - \mathbf{E}^2 \right. \\ &\quad \left. - \mathbf{v}^2 \mathbf{B}^2 - 2\mathbf{E} \cdot (\mathbf{v} \times \mathbf{B})] - 2\gamma^2 (\mathbf{E} \cdot \mathbf{v})^2/c^2 \right\} dV, \end{aligned} \quad (24)$$

$$\begin{aligned} \overset{\text{A}}{\mathbf{p}} &= \frac{1}{\mu\mu_0 c^2} \int_{V'} \mathbf{E} \times \mathbf{B} dV \\ &\quad - \frac{(n^2 - 1)}{\mu\mu_0 c^4} \int_{V'} \left\{ \gamma^2 (\mathbf{E} \cdot \mathbf{v})(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \gamma^4 \mathbf{v} [(\mathbf{E} \cdot \mathbf{v})^2/c^2 \right. \\ &\quad \left. + (\mathbf{B} \cdot \mathbf{v})^2 - \mathbf{E}^2 - \mathbf{v}^2 \mathbf{B}^2 - 2\mathbf{E} \cdot (\mathbf{v} \times \mathbf{B})] \right\} dV. \end{aligned} \quad (25)$$

From (24) and (25) we see that to zeroth order in \mathbf{v} they reduce to the well known expressions for a linear, isotropic and homogeneous medium at rest,

$$\overset{\text{A}}{E}_{(0)} = \frac{1}{2} \int_{V'} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) dV, \quad \overset{\text{A}}{\mathbf{p}}_{(0)} = \frac{1}{c^2} \int_{V'} \mathbf{E} \times \mathbf{H} dV, \quad (26)$$

where we used the constitutive relations in the medium at rest $\mathbf{D} = \epsilon \epsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu\mu_0$.

We consider the special simple case in which the electromagnetic pulse can be approximated by a ‘‘cut’’ plane wave of finite volume $V_p \ll V$, i.e. much less than the volume of the slab, but big enough so that the continuum approximation for the slab is valid, propagating in the direction $\hat{\mathbf{x}}$ of the slab's motion. Solving the macroscopic Maxwell equations inside the medium for light with any polarization (see Appendix A for more details), we can write,

$$\mathbf{B}(x, t) = \frac{1}{v_\beta} \hat{\mathbf{x}} \times \mathbf{E}, \quad (27)$$

where v_β is the phase velocity of light inside the moving medium, given by

$$v_\beta := c \frac{(1 + n\beta)}{(n + \beta)}. \quad (28)$$

Since in our one-dimensional case $\mathbf{v} \perp \mathbf{E}$ and $\mathbf{v} \perp \mathbf{B}$, the expressions (24) and (25) reduce to:

$$\begin{aligned} \overset{A}{E} &= \frac{1}{2\mu\mu_0 c^2} \int_{V'} (\mathbf{E}^2 + c^2 \mathbf{B}^2) dV \\ &+ \frac{(n^2 - 1)}{2\mu\mu_0 c^2} \int_{V'} \gamma^2 (2\gamma^2 - 1) |\mathbf{E} + \mathbf{v} \times \mathbf{B}|^2 dV, \end{aligned} \quad (29)$$

$$\overset{A}{\mathbf{p}} = \frac{1}{\mu\mu_0 c^2} \int_{V'} \mathbf{E} \times \mathbf{B} dV + \frac{(n^2 - 1)}{\mu\mu_0 c^4} \int_{V'} \gamma^4 \mathbf{v} |\mathbf{E} + \mathbf{v} \times \mathbf{B}|^2 dV. \quad (30)$$

If we insert (27) and (28) into (29) and (30), we obtain more compact expressions for $\overset{A}{E}$ and $\overset{A}{\mathbf{p}}$, just in terms of \mathbf{E}^2 :

$$\overset{A}{E} = \frac{1}{\mu\mu_0 c^2} \int_{V'} \frac{n(n + 2\beta + n\beta^2)}{(1 + n\beta)^2} \mathbf{E}^2 dV, \quad (31)$$

$$\overset{A}{\mathbf{p}} = \frac{1}{\mu\mu_0 c^3} \hat{\mathbf{x}} \int_{V'} \frac{n(1 + 2n\beta + \beta^2)}{(1 + n\beta)^2} \mathbf{E}^2 dV. \quad (32)$$

When the pulse is fully inside the slab, we can integrate over the volume V_p of the pulse and the factors with n and β will go out the integral. Therefore, we can relate the Abraham momentum and Abraham energy of the pulse inside the medium by

$$\overset{A}{\mathbf{p}} = \frac{(1 + 2n\beta + \beta^2)}{(n + 2\beta + n\beta^2)} \frac{\overset{A}{E}}{c} \hat{\mathbf{x}}, \quad (33)$$

which is an important result that we will use in the next subsection. It is worthwhile to notice that (33) is valid for any polarization of the “cut” plane-wave pulse, but not for a general pulse form since, if we compare (29) with (30), we see that $\overset{A}{\mathbf{p}}$ and $\overset{A}{E}$ are not proportional in general. As a consistency test, it can be checked that the same result (33) can be obtained if we apply a boost to the well-known Abraham expression (2b) valid in the rest frame of the medium.

4.3. Conservation of the center of energy velocity

In [25], Barnett revitalized the argument of Balazs [23] which states that the conservation of the center of energy velocity, in addition to the conservation of momentum, uniquely selects the momentum of light inside the slab to be the one of Abraham. We will now examine these arguments in more detail. We can check from (33) that when the medium is at rest, we get the typical value of the Abraham momentum, see (2b), $\overset{A}{\mathbf{p}} = (\overset{A}{E}/nc)\hat{\mathbf{x}}$, which we can write in terms of the phase velocity of light in that reference frame $v_0 := c/n$ as

$$\overset{A}{\mathbf{p}} = v_0 \frac{\overset{A}{E}}{c^2} \hat{\mathbf{x}}, \quad (34)$$

i.e. as if it was a particle of “moving mass” $\overset{A}{M} := \overset{A}{E}/c^2$ and velocity $\mathbf{v}_0 = (c/n)\hat{\mathbf{x}}$, in a way similar to (23). In [25, 26] the

explicit definition of the conserved “center of energy velocity” of the system \mathbf{v}_{CE} is not shown, however, from special relativity we know that it is related to the total energy E and the total momentum \mathbf{p} of the system, by

$$\mathbf{v}_{\text{CE}} = \frac{c^2}{E} \mathbf{p}. \quad (35)$$

Now, in order to reproduce Barnett’s argument in detail, we add the momentum of light in the form (34) to the momentum of the slab in (23) and use the total energy and momentum conservation to get

$$\mathbf{v}'_{\text{CE}} = \frac{\overset{A}{E}(c/n) + \overset{m,A}{E} \mathbf{v}}{\overset{A}{E} + \overset{m,A}{E}}. \quad (36)$$

Strictly speaking this argument is *incorrect*, because \mathbf{v}'_{CE} is not a conserved quantity. The expression (34) for the electromagnetic field is only valid when the slab is at rest, but the final velocity of the medium is not zero and the choice of the velocity $v_0 = c/n$ for the light pulse is inappropriate. If we want to write the expression of the momentum of the pulse as if it were a particle with energy $\overset{A}{E}$, then as can be seen from (33), the proper “particle” velocity \mathbf{v}_p should be defined as

$$\mathbf{v}_p := c \frac{(1 + 2n\beta + \beta^2)}{(n + 2\beta + n\beta^2)}, \quad (37)$$

and hence the correct total momentum of the system would have the form $\mathbf{p} = (\overset{A}{E}/c^2)\mathbf{v}_p \hat{\mathbf{x}} + (\overset{m,A}{E}/c^2)\mathbf{v} \hat{\mathbf{x}}$. Together with the total energy of the system $E = \overset{A}{E} + \overset{m,A}{E}$, which is also conserved, the velocity of the center of energy \mathbf{v}_{CE} given by

$$\mathbf{v}_{\text{CE}} = \frac{\overset{A}{E} \mathbf{v}_p + \overset{m,A}{E} \mathbf{v}}{\overset{A}{E} + \overset{m,A}{E}} \hat{\mathbf{x}}, \quad (38)$$

turns out to be a conserved quantity indeed. In general, the center of energy velocity \mathbf{v}_{CE} in (35) is always a conserved quantity in a closed system. Since the total energy-momentum tensor (3) of the closed system is conserved and symmetric, all the components of the total angular 4-momentum tensor $L_{\rho\sigma} := \int_{V_p} (x_\rho T_{\sigma}^0 - x_\sigma T_{\rho}^0)$ are conserved as well, including the components $0a$ associated to the conservation of the velocity of the center of energy. It is important to remark that this conservation holds independently of the choice of the Abraham or the Minkowski momentum to describe the electromagnetic field, because it depends on the total quantities \mathbf{p} and E .

Although formally incorrect, in practice the naive expression $v_0 = c/n$ yields a very good approximation for the particle velocity of the pulse. As we will see in section 4.4, $\beta \sim 10^{-15}$ in a standard case and $\beta \sim 10^{-9}$ in an extreme case, so if we expand (37),

$$\mathbf{v}_p = \frac{c}{n} + 2 \frac{c(n^2 - 1)}{n^2} \beta + O(\beta^2). \quad (39)$$

we see that $\mathbf{v}_p \approx \mathbf{v}_0 = c/n$ is an extremely accurate approximation indeed. It is also remarkable, that the correct velocity that

should enter (38) is also different from the relativistic phase velocity v_β in (28), as one could naively expect as a generalization of v_0 . The non-relativistic expansion of v_β reads,

$$v_\beta = \frac{c}{n} + \frac{c(n^2 - 1)}{n^2} \beta + O(\beta^2), \quad (40)$$

differing from (39) by a factor 2 in the term of first order in β . This term in (40) has been measured with great accuracy in the Fizeau experiment. See, for instance [6], page 187.

4.4. Conservation equations and solution of the slab motion

Since we already know the explicit forms of the energy and momentum of the complete system in the Abraham separation, we can use the two conservation equations to solve the problem of the motion of the slab in terms of the system's parameters. We will consider two states of the system, first when the pulse is traveling with total energy \mathcal{E}_0 in vacuum and the slab of mass M and refraction index n is at rest, and finally when the electromagnetic pulse is completely inside the slab after it already reached a final constant velocity $\mathbf{v} = c\beta\hat{\mathbf{x}}$. Therefore, the energy conservation equation reads

$$E^{\text{m,A}}_{\text{(out)}} + E^{\text{A}}_{\text{(out)}} = E^{\text{m,A}}_{\text{(in)}} + E^{\text{A}}_{\text{(in)}}, \quad (41)$$

$$Mc^2 + \mathcal{E}_0 = \gamma Mc^2 + E^{\text{A}}, \quad (42)$$

and the total momentum conservation in the propagation direction $\hat{\mathbf{x}}$ is given by

$$\mathbf{p}^{\text{m,A}}_{\text{(out)}} + \mathbf{p}^{\text{A}}_{\text{(out)}} = \mathbf{p}^{\text{m,A}}_{\text{(in)}} + \mathbf{p}^{\text{A}}_{\text{(in)}}, \quad (43)$$

$$0 + \frac{\mathcal{E}_0}{c} = \gamma M c \beta + \frac{(1 + 2n\beta + \beta^2) E^{\text{A}}}{(n + 2\beta + n\beta^2) c}. \quad (44)$$

Equations (42) and (44) constitute a system of two equations for two unknowns β and E^{A} , which we can solve in terms of the system parameters \mathcal{E}_0 , M and n . From (42) we can find an expression for E^{A} in terms of β , which reads

$$E^{\text{A}} = \mathcal{E}_0 + Mc^2(1 - \gamma). \quad (45)$$

This last equation already determines that the motion of the slab will be non-relativistic in most practical situations. Let us consider the extreme case when all the energy of the light pulse in vacuum is transformed in kinetic energy of the slab. Then in (45), $E^{\text{A}} = 0$, and we can determine γ_{max} as,

$$\gamma_{\text{max}} = 1 + q, \quad (46)$$

where we have defined the dimensionless parameter q by

$$q := \frac{\mathcal{E}_0}{Mc^2}. \quad (47)$$

The parameter q (together with n) determines the motion of the slab. In practice q is extremely small, as we shall see at the end of the section, of the order $q \sim 10^{-9}$ or less, so from (46) we

see that γ_{max} will be very close to unity and therefore β_{max} will be at most of the order $\beta_{\text{max}} \sim \sqrt{2q} \sim 10^{-4} \ll 1$, resulting in a non-relativistic motion of the slab. Even though we know that the motion will be non-relativistic, we will first present the full equation for β , without any further approximation. Inserting (45) in (44), we get a fourth order polynomial equation for β in terms of the parameters q and n :

$$\begin{aligned} & [(1 + q - nq)^2 + n^2]\beta^4 + [4(1 + q - nq)(n - q + nq) + 2n]\beta^3 \\ & + [2(1 + q - nq)^2 + 4(n - q + nq)^2 + 1 - n^2]\beta^2 \\ & + [4(1 + q - nq)(n - q + nq) - 2n]\beta + [(1 + q - nq)^2 - 1] = 0. \end{aligned} \quad (48)$$

This equation can be solved analytically, but the expression of the solution is large and not instructive. Since we already know that in practice $q \ll 1$ and $\beta \ll 1$, it is interesting to search for a more tractable approximated solution for β . Therefore, if we keep only the first order terms in (48), we get the well known solution of Balazs, Barnett, Mansuripur and many others, for the non-relativistic velocity of the dielectric slab

$$\beta \approx \frac{1}{n}(n - 1)q, \quad (49)$$

or, in more familiar terms,

$$v \approx \frac{(n - 1)}{n} \frac{\mathcal{E}_0}{Mc} > 0, \quad (50)$$

which means that the slab will move in the same direction of the electromagnetic pulse, while the pulse is propagating inside it. If we continue in the non-relativistic limit, the light pulse will spend a time interval $\Delta t \approx nL/c$ inside the slab and therefore its net displacement Δx will be, as usual, $\Delta x \approx (n - 1)L\mathcal{E}_0/Mc^2 > 0$. Additionally, one can also find a solution of (48) which is exact to second order in q :

$$\beta(n, q) = \frac{(n - 1)}{n}q - \frac{(4n^3 - 5n^2 - 2n + 3)}{2n^3}q^2 + O(q^3), \quad (51)$$

where the second term can be considered as the first “relativistic correction” for β . From (51) we can estimate the error that we make by using just the first order non-relativistic approximation (49). Suppose that the slab is made of glass with $n = 1.5$. If it has a mass of the order $M \sim 100g$ and if the light source is a good pulsed laser with energy $\mathcal{E}_0 \sim 1J$, then the parameter q will be typically of the order $q \sim 10^{-15}$ and therefore from (50) we see that $v \sim 0.1 \mu\text{m/s}$. In this case, the difference between the second order solution of (51) and the non-relativistic solution (49) is of the order $\Delta\beta \sim 10^{-31}$ and the relative error of using (49) is $\sim 100q \sim 10^{-13}\%$. In the extreme case of the most powerful lasers available $\mathcal{E}_0 \sim 1kJ$ and a very small dielectric slab with mass $M \sim 10g$, we would in theory be able to achieve a q parameter of the order $q \sim 10^{-9}$ and a final slab velocity of order $v \sim 10 \text{ cm/s}$. In this case, $\Delta\beta \sim 10^{-19}$ and the relative error is $\sim 100q \sim 10^{-7}\%$ and hence we see that the well-known non-relativistic solution (50) is extremely accurate for all practical purposes.

5. Using the Minkowski tensor

5.1. Minkowski energy and momentum expressions

In [6, 7, 16, 19, 23, 25, 28] and other papers, it is argued that the Minkowski momentum fails to describe this slab experiment by predicting the motion of the slab in the opposite direction of the incident electromagnetic pulse. However, the mistake is to consider the same balance equations that are valid for the Abraham momentum, while using the Minkowski momentum for the electromagnetic pulse, thereby tacitly assigning an incorrect energy and momentum to matter in the Minkowski picture. Jones [29] noticed this deficiency and suggested that the correct momentum of matter should include the “forward bodily impulse” the nature of which he was unable to describe. The explanation is simple, though: one needs to use the *canonical momentum* of matter which, combined with the canonical momentum of the electromagnetic field, is conserved in the Minkowski picture.

As we mentioned in section 3, we can formally compute the correct Minkowski quantities from the Abraham expressions for the energy and momentum by adding the proper components of Q_μ^ν . Evaluating the expression for Q_μ^0 in (14), we get

$$Q_\mu^0 = \frac{(n^2 - 1)}{\mu\mu_0 c^2} \frac{n}{(1 + n\beta)^2} E^2(\beta, -1, 0, 0). \quad (52)$$

Then, the Minkowski energy of the pulse is of the form

$$\overset{M}{E} = \overset{A}{E} + \int_{V'} Q_0^0 dV, \quad (53)$$

$$= \frac{n}{\mu\mu_0 c^2} \frac{c}{v_\beta} \int_{V_p} E^2 dV. \quad (54)$$

Using (29) integrated over V_p , we can relate $\overset{M}{E}$ to the Abraham energy of the light pulse by

$$\overset{A}{E} = \frac{(n + 2\beta + n\beta^2)}{(1 + n\beta)(n + \beta)} \overset{M}{E}. \quad (55)$$

These energies coincide, for a given $n \neq 1$, only in the case $\beta = 0$, i.e. in the rest frame of the medium. In the same way, starting from the definitions (15) and (16), and using the expressions (52) and (54), we can determine all the other Minkowski quantities in terms of $\overset{M}{E}$:

$$\overset{M}{\mathbf{P}} = \frac{\overset{M}{E}}{v_\beta} \hat{\mathbf{x}}, \quad (56)$$

$$\overset{m,M}{E} = \gamma M c^2 - \frac{(n^2 - 1)\beta}{(1 + n\beta)(n + \beta)} \overset{M}{E}, \quad (57)$$

$$\overset{m,M}{\mathbf{P}} = \left[\gamma M c \beta - \frac{(n^2 - 1)}{(1 + n\beta)(n + \beta)} \frac{\overset{M}{E}}{c} \right] \hat{\mathbf{x}}. \quad (58)$$

The last term in the *canonical momentum* of the matter (58) accounts for the “forward bodily impulse” of Jones [29].

5.2. Defining a Minkowski velocity

With the results (54), (56), (57) and (58) we can express the center of energy velocity in the Minkowski picture as

$$v_{CE} = \frac{c^2}{E} \left[\gamma M c \beta - \frac{(n^2 - 1)}{(1 + n\beta)(n + \beta)} \frac{\overset{M}{E}}{c} + \frac{\overset{M}{E}}{v_\beta} \right] \quad (59)$$

$$= \frac{c^2}{E} \left[v \frac{\gamma M c^2}{c^2} + \frac{c(1 + 2n\beta + \beta^2)}{(1 + n\beta)(n + \beta)} \frac{\overset{M}{E}}{c^2} \right] \quad (60)$$

$$= \frac{v(\gamma M c^2) + v_M \overset{M}{E}}{E}, \quad (61)$$

where we defined the Minkowski velocity v_M of the field as

$$v_M := c \frac{(1 + 2n\beta + \beta^2)}{(1 + n\beta)(n + \beta)}. \quad (62)$$

To get the velocity (62), we shifted terms from $\overset{m,M}{P}$ to $\overset{M}{P}$ and therefore it does not satisfy the prescription of a “particle velocity”,

$$v_{em} = c^2 \frac{\mathbf{P}_{em}}{E_{em}}, \quad v_{mat} = c^2 \frac{\mathbf{P}_{mat}}{E_{mat}}, \quad (63)$$

as for the case of Abraham light and matter velocities (23) and (37). In fact, v_M in (62) is a very artificial velocity that one would have to associate to the field with the Minkowski energy, in order to keep the velocity v for the block, and still obtain the correct (relativistic) center of energy velocity (38). If we expand (62) in powers of β we see that v_M is also similar to all the other previously defined velocities,

$$v_{p,M} = \frac{c}{n} + \frac{c(n^2 - 1)}{n^2} \beta + O(\beta^2), \quad (64)$$

and hence equals c/n in the non-relativistic limit to zeroth order in β , as well as the Abraham particle velocity. We could also define a Minkowski “particle” velocity following the prescription (63). Therefore, using (56), (57) and (58), we have

$$v_{p,M} := \frac{c^2 \overset{M}{P}}{\overset{M}{E}} = \frac{c^2}{v_\beta} = c \frac{(n + \beta)}{(1 + n\beta)}, \quad (65)$$

$$v_{mat,M} := \frac{c^2 \overset{m,M}{P}}{\overset{m,M}{E}} = c \frac{\left[\gamma M c^2 (1 + n\beta)(n + \beta) \beta - (n^2 - 1) \overset{M}{E} \right]}{\left[\gamma M c^2 (1 + n\beta)(n + \beta) - (n^2 - 1) \beta \overset{M}{E} \right]}. \quad (66)$$

With (65) and (66) the expression for the velocity of “center of energy” naturally assumes the form $v_{CE} = (\sum_i v_i \cdot E_i) / (\sum_i E_i)$, because it depends only on the total quantities, *but neither (65) nor (66) coincide with a velocity of the system which one can easily identify and interpret* (like the velocity of the block v or the phase velocity of the field v_β , for example). Indeed, $v_{mat,M}$ in (66) depends on the energy of the pulse, which is very counter-intuitive. When there is no light pulse, i.e. $\overset{M}{E} = 0$ and $v_{mat,M}$ reduces to the velocity of the block v .

5.3. Minkowski balance equations

Since we already know all the explicit expressions for the Minkowski momentum and energy of the field and slab, we can use them to write the balance equations and correctly solve for the slab velocity also with the Minkowski formulation. From (56), (57) and (58), we have

$$Mc^2 + \mathcal{E}_0 = \left[\gamma Mc^2 - \frac{(n^2 - 1)\beta}{(1 + n\beta)(n + \beta)} \frac{M}{E} \right] + \frac{M}{E}, \quad (67)$$

$$\frac{\mathcal{E}_0}{c} = \left[\gamma Mc\beta - \frac{(n^2 - 1)}{(1 + n\beta)(n + \beta)} \frac{M}{c} \right] + \frac{(n + \beta)}{(1 + n\beta)} \frac{M}{c}. \quad (68)$$

Taking (67) and dividing it by Mc^2 , we get

$$q_M = \frac{(1 + n\beta)(n + \beta)}{(n + 2\beta + n\beta^2)}(q + 1 - \gamma), \quad (69)$$

where $q_M := \frac{M}{Mc^2}$, following the definition (47). Then, if we divide (68) by Mc and use (69), we obtain after some algebra the same fourth order equation for β in (48), which we obtained with the Abraham formulation. Therefore in the the Minkowski picture, we obtain the same solution $\beta = \beta(n, q)$ for the motion of the slab. The authors who claim that the Minkowski momentum is unable to describe the slab plus light pulse system use the equations (67) and (68), but without the second terms inside the bracket on the r.h.s. and hence they use incorrect balance equations.

6. Conclusions

The fundamental equations, which govern the interactions and motion within the total system of electromagnetic field plus a material medium, are the macroscopic Maxwell equations, the constitutive relations and the hydrodynamic equations for the fluid. The energy and momentum balance equations for the total system have a clear physical meaning and we can use them, as here explicitly shown, to determine the dynamics of the system. The Abraham and Minkowski momenta can be understood as different separations of the total momentum, a choice of which does not affect the physical predictions.

When we use the total energy-momentum tensor, the conservation of momentum and of the velocity of “center of energy” is always satisfied for any specific separation, because it involves only the total quantities. However, when we use the “Abraham separation”, we assign the energy-momentum tensor of a perfect fluid to the material subsystem, as if it were in isolation. As a result, the definition (63) of the Abraham velocity of matter coincides with the velocity of the block and it is the only separation in which this happens. This fact explains why the Abraham tensor is relevant for the case of an homogeneous and isotropic medium, and for the Einstein box theories in particular. Since in this picture we can consistently interpret the block as a particle, the remaining term can be naturally interpreted as the momentum of a “light particle” with velocity v_p given in (37), which in the non-relativistic approximation (very accurate for these cases, as we have demonstrated) coincides with the

phase velocity of light when the medium is at rest $v_0 = c/n$. In other words, with the “Abraham separation” the property of inertia of energy is not only satisfied in the total system, as special relativity requires, but also in each subsystem separately. This was already noticed by Brevik in 1979, see [6], page 192.

In our opinion, these are the best arguments which support the usefulness of the interpretation of the Abraham momentum as the “kinetic momentum” of the field when the momentum of light is introduced in the usual non-relativistic “ mv ” form,

$$\overset{A}{p} := \overset{A}{M}v_0 = \frac{\overset{A}{E}}{c^2} \frac{c}{n} = \frac{1}{n} \frac{\overset{A}{E}}{c}. \quad (70)$$

At the same time, in spite of the fact that the Abraham choice is simpler than Minkowski’s one for the case of the block, our analysis in section 5 clearly demonstrates that the Minkowski definition is also perfectly consistent for the Einstein box experiment, contrary to what is sometimes claimed [6, 7, 16, 19, 23, 25, 28], provided one considers the correct Minkowski expressions for the energy and momentum of matter. For a complementary discussion considering other explicit field configurations, see [30]. Again, only the total quantities are relevant for the description of the system.

In the nonrelativistic discussion of the balance equations for the slab, see for instance [23, 25, 26, 27], the Abraham momentum is selected as a consequence of treating the contribution of the light pulse to the velocity of the center of energy as if it were a particle moving with the phase velocity of the wave. We want to stress that this choice is not justified from the point of view of field theory. Additionally, our fully relativistic analysis shows that this assumption would only be consistent with the global conservation laws of the total system if one introduces suitable (ad hoc) “particle velocities” for the pulse, both in the Minkowski and Abraham pictures. However, these velocities do not correspond in general to any well defined velocity in the system. In particular, they do not coincide with the phase velocity of the wave in the moving medium, which is only true to zeroth order in the final slab velocity.

The Abraham tensor is useful for isotropic media [3], in the sense that if the block is described by a dust energy-momentum tensor, all other terms are contained in the Abraham tensor. However, this may not be the case in general media, [3]. Therefore it is necessary to extend the analysis and study more complex media, for instance with anisotropic or magneto-electric properties. In any case, our analysis, along with those in ref. [3, 5, 9, 11], shows that the Abraham choice of the “correct” momentum of a light pulse is only one possibility, simple and useful for the description of isotropic media, but not at all an unique one.

Appendix A. Electromagnetic plane wave solution in a linear, homogeneous, isotropic, moving medium

The macroscopic Maxwell equations and the constitutive relation for a *linear* medium are given in covariant form by [3]

$$\partial_\nu H^{\mu\nu} = J_{\text{ext}}^\mu, \quad (\text{A.1})$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0, \quad (\text{A.2})$$

$$H^{\mu\nu} = \frac{1}{2} \chi^{\mu\nu\rho\sigma} F_{\rho\sigma}. \quad (\text{A.3})$$

In particular, we consider an *homogeneous, isotropic, non-dissipative* and *non-dispersive* medium in motion and therefore we can express the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ in terms of the well-known Gordon optical metric $\gamma^{\mu\nu}$ by

$$\chi^{\mu\nu\rho\sigma} := \frac{1}{\mu_0\mu} (\gamma^{\mu\rho}\gamma^{\nu\sigma} - \gamma^{\mu\sigma}\gamma^{\nu\rho}), \quad (\text{A.4})$$

where,

$$\gamma^{\mu\nu} := g^{\mu\nu} + \frac{(n^2 - 1)}{c^2} u^\mu u^\nu, \quad (\text{A.5})$$

as it was first derived by Gordon in 1923 [31]. The Minkowski metric $g_{\mu\nu}$ is defined as $g_{\mu\nu} = \text{diag}(c^2, -1, -1, -1)$, ε is the dielectric constant, μ the relative permeability, $n := \sqrt{\varepsilon\mu}$ the refractive index, u^μ the 4-velocity of the moving medium and $c := 1/\sqrt{\mu_0\varepsilon_0}$ the velocity of light in vacuum. If we look for plane wave solutions of the form

$$F_{\mu\nu} = \Re \left\{ \tilde{F}_{\mu\nu} e^{ik_\lambda x^\lambda} \right\}, \quad (\text{A.6})$$

then the optical metric $\gamma^{\mu\nu}$ determines the dispersion relation:

$$\gamma^{\mu\nu} k_\mu k_\nu = 0. \quad (\text{A.7})$$

We consider the one-dimensional slab problem for which k_μ and u^μ are of the form

$$u^\mu = (\gamma, \gamma v, 0, 0), \quad k_\mu = (\omega, -k, 0, 0), \quad (\text{A.8})$$

where v is the velocity of the slab and k the wave number. Using (A.8a) in (A.5), the explicit expression for $\gamma^{\mu\nu}$ reads,

$$\gamma^{\mu\nu} = \begin{pmatrix} [1 + (n^2 - 1)\gamma^2]/c^2 & (n^2 - 1)\gamma^2\beta/c & 0 & 0 \\ (n^2 - 1)\gamma^2\beta/c & -1 + (n^2 - 1)\gamma^2\beta^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.9})$$

Then, using (A.8b) and (A.9) in (A.7) and solving for ω in terms of β , n and k we obtain the dispersion relation of light inside the moving medium, $\omega(k) = v_\beta k$, where v_β is the phase velocity of the waves inside the moving medium, given by (28).

Finally, inserting the ansatz (A.6) into the Maxwell equations (A.1) and (A.2), and using (A.4), (A.5), one can find, after some algebra, general explicit expressions for the field strengths $\mathbf{E}(x, t)$ and $\mathbf{B}(x, t)$. However, here we just display the properties that are used in the main text, namely $\hat{\mathbf{x}} \cdot \mathbf{E}(x, t) = 0$ and (27). Relations (28) and (27) can be also obtained by applying

the appropriate relativistic transformation of the fields from the rest frame of the medium. Notice however that the relativistic transformation of the phase velocity is in general different from the usual particle velocity transformation, but they do coincide if the phase velocity is parallel to relative velocity. See, for instance, [32] and [33].

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